





GEOMETRIC FUNCTIONS  
IN COMPUTER AIDED GEOMETRIC DESIGN

---

OSCAR RUIZ  
CARLOS CADAVID



Ruiz, Oscar

Geometric functions in computer aided geometric design /

Oscar Ruiz, Carlos Cadavid. -- Medellín : Fondo Editorial

Universidad EAFIT, 2008.

140 p. : il. ; 24 cm. -- (Colección académica)

ISBN 978-958-720-016-4

1. Diseño con ayuda de computador 2. Geometría con ayuda de computador 3. Gráficos por computador 4. Ingeniería de diseño I. Cadavid, Carlos, 1955- II. Tít. III. Serie. 620.00420285 cd 21 ed.

A1184811

CEP-Banco de la República-Biblioteca Luis Ángel Arango

## GEOMETRIC FUNCTIONS IN COMPUTER AIDED GEOMETRIC DESIGN

Primera edición: septiembre de 2008

Sexta reimpresión: julio de 2017

© Óscar Ruiz

© Carlos Cadavid

© Fondo Editorial Universidad EAFIT

Cra.49 No. 7 sur-50

[www.eafit.edu.co/fondoeditorial](http://www.eafit.edu.co/fondoeditorial)

Email: [fonedit@eafit.edu.co](mailto:fonedit@eafit.edu.co)

ISBN: 978-958-720-016-4

Imagen de carátula:

COVER

Aphrodite's mesh was ray-traced using a simulated screen shaped into a helicoid tape embedded in a 3D ellipsoid. Prof. Dr. Eng. Oscar Ruiz and undergraduate assistants Carlos Vanegas and Ricardo Serrano. CAD CAM CAE Laboratory, EAFIT University, 2006-2007.

Editado en Medellín, Colombia

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Basic Concepts. Groups</b>	<b>3</b>
2.1	Functions . . . . .	3
2.1.1	Properties of Functions . . . . .	3
2.1.2	Composition of Functions . . . . .	4
2.2	Binary Operations and Groups . . . . .	5
2.2.1	Binary Operations . . . . .	5
2.2.2	Groups . . . . .	6
2.3	Square Matrices . . . . .	8
2.3.1	Matrix Invertibility . . . . .	9
2.3.2	Properties of the Inverse and Transposed Matrix	9

2.4	The General Linear Group $GL(n, R)$ . . . . .	9
2.5	The Positive Linear Group ( $GL^+(n, R)$ ) . . . . .	10
2.6	The Orthogonal Group $O(n)$ . . . . .	11
2.7	The Special Orthogonal Group $SO(n)$ . . . . .	12
2.8	The Group $(X, \circ)$ . . . . .	13
2.9	Transformations . . . . .	14
2.9.1	Proposed Exercise . . . . .	15
2.9.2	Linear Transformations . . . . .	15
2.10	Affine Transformations . . . . .	16
2.10.1	Affine Transformations in $R^2$ . . . . .	17
2.10.2	Proposed Exercise . . . . .	18
2.11	Summary . . . . .	19
<b>3</b>	<b>Property Invariance under Geometric Transformations</b>	<b>21</b>
3.1	Introduction . . . . .	21
3.2	Property Preservation in General Transformations . . . . .	23
3.2.1	Colinearity Preservation . . . . .	24
3.2.2	Distance Preservation . . . . .	24
3.2.3	Volume Preservation . . . . .	24

3.2.4	Angle Preservation . . . . .	26
3.2.5	Orientation Preservation . . . . .	32
3.2.6	Origin Preservation . . . . .	32
3.2.7	Example. Plane Reflection . . . . .	32
3.3	Property Preservation in Affine Functions . . . . .	33
3.3.1	Discussion. Affine and Linear Transformations in 3D . . . . .	34
3.4	Example. A Linear Transformation in $R^2$ . . . . .	35
3.5	Example. Non-Linear Transformations $R^2 \rightarrow R^2$ . . . . .	37
3.5.1	Solution . . . . .	38
3.6	Proposed Exercise. Non-Linear Transformations $R^2 \rightarrow$ $R^2$ . . . . .	39
3.6.1	Area-Preservation. Proof . . . . .	39
3.6.2	Area-Preservation. Programming . . . . .	40
3.7	Proposed Exercise. Affine Transformations $R^2 \rightarrow R^2$ , $Aff(2, R)$ . . . . .	40
3.8	Solved Exercise. Non-affine Transformations $R^2 \rightarrow R^2$ .	41
3.8.1	Solution . . . . .	41

3.9	Homogeneous Coordinates . . . . .	45
3.9.1	Definition . . . . .	45
3.9.2	Rationale for Homogeneous Coordinates . . . . .	46
3.9.3	Transformations in Homogeneous Coordinates . . . . .	48
3.9.4	Proposed Exercises . . . . .	49
3.10	Coordinate Systems . . . . .	50
3.10.1	Definition. Coordinate Systems . . . . .	50
3.10.2	Definition. Right Handed Canonical Coordinate System in $R^3$ . . . . .	50
3.10.3	Proposed Exercise . . . . .	50
3.10.4	Solved Exercise . . . . .	51
<b>4</b>	<b>Rigid Transformations in <math>R^3</math></b>	<b>55</b>
4.1	Definition. Rigid Transformations . . . . .	55
4.2	Pure Translations . . . . .	55
4.3	Pure Rotations . . . . .	57
4.3.1	Rotations about the Principal Axes . . . . .	57
4.3.2	Proposed Exercises . . . . .	60
4.4	Eigenvalues and eigenvectors of $R \in SO(3)$ . . . . .	61



4.4.1	Eigenvalues and Eigenvector of Matrices	
	$SO(3)$ . . . . .	61
4.4.2	Transformation Sequences . . . . .	62
4.4.3	Solved Exercise. Transformation Sequences . . .	62
4.4.4	Proposed Exercises. Rotations about Main Axis	69
4.4.5	Rotations about Arbitrary Axis. Quaternion . .	69
4.5	General Rigid Transformation Using Quaternions . . .	71
4.6	Solved and Proposed Exercises . . . . .	72
4.6.1	Solved Exercise. Quaternion . . . . .	72
4.6.2	Proposed Exercise. Rigid Transformations . . .	76
4.6.3	Proposed Exercise. Flight Simulator . . . . .	79
<b>5</b>	<b>Non-Rigid Transformations and Functions</b>	<b>83</b>
5.1	Non-Rigid Affine Transformations . . . . .	84
5.1.1	Scalings . . . . .	84
5.1.2	Reflections . . . . .	87
5.1.3	Shears . . . . .	94
5.2	Pseudo-affine Geometric Functions. Parallel Projections	97
5.2.1	Orthogonal Parallel Projections . . . . .	98

5.2.2	Non-orthogonal Parallel Projections . . . . .	100
5.3	Non-Linear Non-Invertible Functions. Perspective Pro- jections . . . . .	106
5.3.1	Perspective of a Point . . . . .	107
5.3.2	Perspective of a Line . . . . .	108
5.3.3	Perspective and Partition of the Lines in $R^3$ . .	113
5.3.4	Proposed Exercise. Perspective Projection . . .	119

# Chapter 1

## Introduction

There are plenty of books that treat the topic of Computer Aided Geometric Design and Applications. These books usually have either one of two extreme approaches: purely operative and purely mathematical. The present book intends to fill this gap, by discussing important underlying mathematical facts and terminology of Computer Aided Geometric Design, while at the same time giving the reader a direct insight in the practical consequences of such facts and terms.

The overall content of the undergraduate courses Introduction to CAD CAM Systems, Introduction to Computer Aided Geometric Design or equivalent spans (i) Geometric Transformations, (ii) Parametric Curves and Surfaces, and (iii) Geometric and Solid Modeling. The present book deals with Geometric Transformations. The authors found that a previous material by them ([Rui05]) presents the three topics in a very practical manner, leaving aside mathematical foundations that may be useful for the educated CAGD developer. The present book intends to represent such a deeper view, in the specific realm of Geometric Transformations.

The term Geometric *Transformations* is widely used in the practice of Computer Aided Geometric Design and Manufacturing. However, it is, formally, incorrect, since the material covers functions that are not *transformations* in the mathematical sense. They are not bijections. But they are useful, and necessary (an example are projections). The authors do not intend to solve a very ancient terminology issue. Therefore, they simply go along with the term, and write about Geometric Transformations referring to rotations, translations, perspectives, shears, affine, etc.

The material included in these notes corresponds to the lectures of the course IM024, Introduction to CAD CAM Systems, delivered by Profs. Oscar Ruiz and Carlos Cadavid in EAFIT University in the period 1996-2008. The exercises were assigned by Profs. Ruiz and Cadavid, and its solution and documentation supervised by them, as appears in the original manuscripts. At the end of the book a list of undergraduate students is given, who carried out such academic and editorial work under the supervision of Profs. Ruiz and Cadavid.

Chapter 2 explores the basic terminology required, and the concept of (affine) groups. Chapter 3 discusses the effects of the terms just introduced, on the property invariance of transformations. Chapter 4 is devoted to the particular domain of Rigid Transformations, fundamental for kinematics, robotics, dynamic simulations, computer graphics, etc. Chapter 5 deals with non rigid transformations, such as parallel and perspective projections.

The reproduction of this material is forbidden without the express written consent of the authors and their sponsoring institutions.