A Course of Plane Geometry

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Mathematics, rightly viewed, possesses not only truth,
but supreme beauty–a beauty cold and austere, like that of sculpture,
without appeal to any part of our weaker nature,
without the gorgeous trappings of painting or music,
yet sublimely pure, and capable of a stern perfection
such as only the greatest art can show.

Bertrand Russell

This book is dedicated, with all my love, to Adriana and Simón, and to my father, Alberto, who taught me the supreme value of the intangible, and to my mother, Leonor, from whom I got the enjoyment of clarity and reason.

Preface

In 2010 I was asked to teach the course *Geometry in Context*, a first semester course for students of the programs of Mathematical Engineering and Physical Engineering, at Universidad EAFIT, in Medellín, Colombia. The goal of the course was that students learned euclidean plane geometry in the classical way, and also that they applied this theoretical knowledge in a real situation, by requiring them to build a mechanism of their choice, which applied some geometrical principle. In preparing the course I consulted several books presenting euclidean plane geometry in the classical way. In studying these books I started to feel uneasy, because the discussion did not adhere to the modern standard of rigor, making proofs kind of difficult to follow, at least to me. I was aware that the great mathematician David Hilbert had proposed a rigorous presentation of the euclidean geometry of space in his book *Grundlagen der Geometrie* (Foundations of Geometry)[4], published in 1899. I went to our university's library, looking for a copy of Hilbert's book. Old books are usually hard to read, and Hilbert's book was in that category. I kept looking for a modern book which explained Hilbert's work more clearly, and I stumbled upon the book *Geometry: Euclid and Beyond* [1], by R. Hartshorne. I browsed the book and I knew I had found the reference I needed. The first chapter of the book presents classical euclidean geometry pointing out its shortcomings, and the second chapter presents Hilbert's axiomatization of (plane) geometry and discusses how to interpret and prove the propositions in Book I of the Elements, in the light of Hilbert's approach. When I began reading chapter two more carefully I was amused when I found out that Hartshorne's presentation made it possible to introduce, *in the most natural way*, non-euclidean geometries from the very beginning! A subject that is usually taught in a Differential Geometry course. But I soon realized that Hartshorne's book is rather dense, going too fast for a first semester student. My task in teaching the course was to explain slowly and in complete detail, most of chapter two of Hartshorne's book. This book is an account of this effort. *I follow Hartshorne's approach throughout*, and I even paraphrase some of the interesting problems he proposes.

Acknowledgments

I want to warmly thank those who helped me, in one way or another, in this enterprise. To Professor Jay Jorgenson, for his friendship and encouragement, and for accepting being this work's supervisor; to my student Juan Fernando García, for reading the entire book to make sure that there were not any logical mistakes, and for transforming my sketchy pictures into the nice and detailed pictures illustrating this book; to my colleague, María Eugenia Puerta, for her important suggestions as to how to make the treatment of certain matters more accesible to the readers; to my colleague Gabriel Loaiza, for suggesting interesting new exercises; to my students of Mathematical Engineering and Physical Engineering, whose comments have helped me over the years to improve my explanations; and to Universidad EAFIT, for allowing me to dedicate the second semester of 2016 to the writing of this book.

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Introduction

1.1 A Short History of Geometry

It is safe to say that the first geometric facts recorded in human history are found within the Egyptian and the Babylonian civilizations. There is strong evidence suggesting that even the Pythagorean Theorem was well known to these civilizations. However, these discoveries were only empirical facts, geometrical regularities that seemed to occur in every case considered. From this evidence, they would come to believe that these were universally true statements, although it seems that nobody bothered to find out why these phenomena took place, or how to "prove" that they were indeed valid in any case. I was not until the Greeks that mathematicians discovered a trustworthy method to know *for sure* the validity or falsity of any given geometric statement. The method, known today as the *axiomatic method*, consists in first taking certain geometrical facts, called axioms, or postulates, or principles, as self-evident, and then, based only on them, and by means of pure reasoning, to derive any other geometrical truth. This is one of the most important inventions of humankind. It *initiates mathematics* as we understand it today, and provides the paradigm for *half* the scientific method, which is nothing else but the addition of experimentation to the axiomatic methodology.

Within the Greek world, the peak of maturity of the axiomatic method was attained with the “publication” of Euclid’s *Elements*. Euclid lived approximately between the middle of the fourth century B.C. and the middle of the third century B.C., mainly in Alexandria, in the Hellenistic part of Egypt. The *Elements* is a collection of thirteen books, containing an axiomatic development of plane and space geometry, elementary number theory and incommensurable lines. Until the beginning of the twentieth century, the *Elements* was the main textbook for teaching mathematics, especially geometry.

After its publication, various authors detected two weak points in Euclid’s work: The feeling that the fifth postulate was not as self-evident as the previous four, and that it should be derived from them; and, secondly, the occasional departure from modern standards of rigor, and even from his own standards.

The first weak point was detected almost immediately. It is even believed that Euclid himself regarded the fifth postulate as different from the other four, in that it was not as self-evident. He was probably forced to add it when he realized that certain propositions towards the end of Book I of the Elements could not be proved without it. Throughout history, most scholars who attempted to remedy this situation followed one of the following two strategies: i) they struggled to “prove” the fifth postulate using only the first four; ii) they attempted to introduce a new postulate that seemed more self-evident than the fifth, and from which, in addition to the other four, they could derive it. Ptolemy (100 A.D.-170 A.D., Alexandria), Proclus (410 A.D.-485 A.D., Athens), Ibn al-Haytham (965 A.D.-1039 A.D., Cairo), Nasir al-Din al-Tusi (1201 A.D.-1274 A.D., Persia), Sadr al-Din (son of Nasir al-Din al-Tusi), Giordano Vitale (1610 A.D.-1711 A.D., Italy), Girolamo Saccheri (1667 A.D.-1733 A.D., Italy), Johann Lambert (1728 A.D.-1777 A.D., Switzerland), are the most eminent followers of the first approach. Omar Khayyám (1050 A.D.-1123 A.D., Persia), John Playfair (1748 A.D.-1819 A.D., Scotland) are among the most famous mathematicians who adopted the second approach.

The "proofs" provided by the ones who followed approach i) were subsequently shown to be wrong, usually because their authors had *unconsciously* used an "obvious" fact which turned out to be *equivalent* to the fifth postulate. This makes their arguments ultimately dependant on the fifth postulate itself.

Followers of approach ii) never succeeded in finding a postulate as self-evident as the first four from which they could derive Euclid's fifth axiom. Many authors did find postulates with this property, but as non self-evident as the fifth.

This state of affairs changed abruptly in the first half of the nineteenth century with the independent realization by Gauss (1817), Lobachevsky (1829), and Bolyai (1831), of the existence of geometries satisfying the first four postulates but not satisfying the fifth. The existence of such geometries constitutes irrefutable proof that the fifth postulate cannot be derived from the first four, in other words, that the fifth postulate is *independent* from the other axioms. This is considered *one of the most important scientific discoveries of all time*, having a profound impact in our understanding of how the human mind apprehends reality. In particular, it made evident the distinction between formal discourse (theory) and the objects it intends to describe (models), starting the development of one of the central branches of mathematical logic, known today as *Model Theory*. The discovery of non-euclidean geometries, together with the work of Gauss on curved surfaces, initiated a process, mainly led by the great german mathematician Bernhard Riemann, that vastly generalized the subject of geometry, by defining Riemannian Manifolds, and regarding them as the central object of study in geometry. This development constituted the mathematical framework for the formulation of Einstein's Theory of General Relativity where Space-Time is actually conceived as a Pseudo-Riemannian Manifold, a slight variation of Riemann's original concept.

Let us now talk about the other weak point found in Euclid's work, namely the occasional departure from modern standards of rigor, and even from his own standards. This criticism started with the revision of the foundations of geometry motivated by the discovery of non-euclidean geometries. The criticism was centered around the following issues:

1. Lack of recognition of the necessity of having primitive terms, i.e., objects and notions that must be left undefined.

2. The use of the "superposition method" without any axioms backing it up.

3. Lack of a concept of continuity needed to prove the existence of some points and lines that Euclid constructs. This happens already when proving Proposition 1 of Book I!

4. Lack of clarity on whether a straight line is infinite or boundaryless in the second postulate.

5. Lack of the concept of *betweenness*, making some arguments depend on the figure.

Different authors have found different ways to remedy this situation. Like David Hilbert, by rigorously filling in the gaps in Euclid's work; some others, like George David Birkhoff, by entirely remodelling the theory, formulating axioms around different concepts.

Let us consider Hilbert's approach. In 1899 Hilbert published his book "Grundlagen der Geometrie" (*The Foundations of Geometry*). In this book, he proposes an axiomatic system for solid geometry, one from which every theorem can be derived by following a strict sequence of rules of inference, starting from a fixed set of formal assumptions stripped of any intuitive content. For Hilbert, relying on figures, using any intuitions about the nature of geometric objects, or introducing any extra assumptions lying beyond the strict syntactical concepts, is completely ruled out. The book has figures, but they are only used as a heuristic guide, and could be dispensed of without affecting the content of the book. *Grundlagen der Geometrie* presented geometry for the first time in history, in a *purely formal way*, i.e., in which the meaning given to the objects in question plays no role whatsoever. The only place where intuition plays a role is in the choice of the axioms themselves. Once the axioms are chosen, the original meaning of the objects can be forgotten without compromising in the least the development of the theory. It can be said that Hilbert presents solid geometry so that *it can be understood by lawyers* (no offense intended), in that it is not necessary to associate geometrical images to the discourse, because the discourse is authentically independent of any interpretation. Hilbert presented his axiomatic system in groups of axioms, each group concerning an aspect of solid geometry. Although Hilbert's axioms formalize solid geometry, it is possible to extract from it a subset of axioms for plane geometry. The first group is formed by eight axioms, the so called *Axioms of Incidence*, which capture the laws governing the incidence relations between points, lines and planes in space. Only three of them are necessary for doing plane geometry. The second group is formed by four axioms called Axioms of Order (or Betweenness). These govern the behaviour of the intuitive notion that a point lies between two other points. The four of them are necessary for doing plane geometry. The third group is formed by six axioms, the Axioms of Congruence, which capture the laws governing the behaviour of congruence of segments and congruence of angles. These six axioms are necessary for developing plane geometry.

For Hilbert's program, the main goal is not only the formalization of Euclidean geometry, but of mathematics as a whole. For him, the most important problem of all mathematics was the foundation of mathematics itself on a solid basis. This meant to Hilbert to reconstruct his own discipline as a purely formal science. This is known as *Hilbert's Formalization Program.* He dreamed of presenting all of Mathematics in the same way he had presented Solid Geometry. Any formal system, as Hilbert envisioned it, must have two fundamental properties: *Consistency and Completeness.* Consistency means that it is not possible to derive within the theory some statement P and its negation. Completeness, on the other hand, means that for each statement P expressible within the system, either P or its negation $\neg P$ can always be derived. Consequently, a formal system is Consistent and Complete if for each statement P expressible within the system, either P or $\neg P$ can be derived from the axioms, but not both. In his *Grundlagen der Geometrie*, Hilbert proves both the consistency of this axiomatization, and the nonredundancy of the axioms, by constructing models of his system.

1.2 What you will and will not learn in this book

Although this is a book about plane geometry, it only contains very basic results. The most sophisticated results appear in the last chapter, in which many of the propositions of Book I of Euclid's Elements are proved. For example, you will not find any mention of the Pythagorean theorem. The emphasis is in the rigorous development of the material, following Hilbert's axiomatic system. Many results are presented which are "intuitively obvious", and whose proofs are rather involved, pointing out the price one has to pay for deriving everything from the axioms through pure reasoning. In this book you will also learn about plane non-euclidean geometry from the very beginning. This is made possible by the method adopted of thinking of plane geometries as set theoretical structures in which a certain collections of axioms hold.

1.3 Audience prerequisites and style of explanation

This book is essentially self-contained. The only previous knowledge required is high school algebra and the understanding that usual algebraic rules for transforming expressions, and solving equations and inequalities, can actually be derived from the properties of addition, multiplication, exponentiation and order in the real number system. Another prerequisite is of psychological nature: the reader is expected to find delight in rigorous thinking. This is absolutely necessary to enjoy the book. We warn the reader that due to the fact that matters are treated with complete rigor, the reading of arguments quite often may turn painful.

It is important to remark that a deliberate effort was made in presenting algebraic manipulations by what they are, i.e. logical transformations of statements. Let us consider for example the solution process of the equation $5x - 2 = 2x + 7$ in the real number system. In high school this process is explained as follows:

"Let us solve the equation $5x - 2 = 2x + 7$. The -2 passes to the other side as $+2$, and the $2x$ passes to the other side as $-2x$, obtaining $5x - 2x = 7 + 2$, which is $3x = 9$. Now since 3 is multiplying on the left hand side, it passes to divide to the right hand side, and so $x = \frac{9}{3}$. In this way we see that $x = 3$".

This is not an explanation at all! This is the application of an algorithm which indeed solves the equation. A good explanation would be as follows:

"Let us solve the equation $5x-2 = 2x+7$ in the real number system. This means that we want to determine all the possible real numbers x such that five times x minus 2 equals twice x plus 7. Properties of addition and multiplication among real numbers, imply the validity of all the following assertions.

The sentence

"x is a real number such that $5x - 2 = 2x + 7$"

is logically equivalent to the sentence

"x is a real number such that $(5x - 2) + 2 = (2x + 7) + 2$"

(Logical equivalence means that the first sentence implies the second sentence, and that the second sentence implies the first sentence).

Likewise, the sentence

"x is a real number such that $(5x - 2) + 2 = (2x + 7) + 2$"

is logically equivalent to the sentence

"x is a real number such that $5x = 2x + 9$".

Now, the sentence

"x is a real number such that $5x = 2x + 9$"

is logically equivalent to the sentence

$$\text{“}x \text{ is a real number such that } 5x - 2x = (2x + 9) - 2x\text{”}$$

and this last sentence is logically equivalent to the sentence

$$\text{“}x \text{ is a real number such that } 3x = 9\text{”}.$$

Finally, the sentence

$$\text{“}x \text{ is a real number such that } 3x = 9\text{”}$$

is logically equivalent to the sentence

$$\text{“}x \text{ is a real number such that } \frac{3x}{3} = \frac{9}{3}\text{”}$$

which is logically equivalent to the sentence

$$\text{“}x \text{ is a real number such that } x = 3\text{”}.$$

In conclusion, the sentence

$$\text{“}x \text{ is a real number such that } 5x - 2 = 2x + 7\text{”}$$

is logically equivalent to the sentence

$$\text{“}x \text{ is a real number such that } x = 3\text{”}.$$

But determining all the possible objects x satisfying the latter condition is trivial; only the real number 3 satisfies it. One concludes that $x = 3$ and no other real number, is such that $5x - 2 = 2x + 7$.”

We make some remarks about the exercises proposed in the book. They vary in several respects. Some exercises are proposed as the theory is developed. These type of exercises are meant to help understanding the ideas that are being developed. At the end of some sections or strings of sections there are sets of exercises. These exercises are intended to expose the reader to variations of the situations treated in the corresponding section or string of sections. Many times in reading a proof, the reader will find indications like (?), (do it!), (why?), (check!), inviting the reader to reflect or take the corresponding action, about what has just been claimed. Also, some parts of some proofs and examples, are explicitly left as exercises for the reader.

Finally, the book has many, many pictures, for illustrating concepts, steps of proofs, etc. There is a constant effort in presenting two pictures of the same concept, an abstract one and a concrete one, where the plane is taken to be the usual euclidean plane.

1.4 Book plan

Chapter 2 is a preliminary chapter, necessary for the understanding of the rest of the book. It starts with a review of the methods for proving statements of the form "P implies Q", and also of methods for proving other types of statements, with particular emphasis on the Induction Method, used for proving statements of the form "$P(n)$ for $n \geqslant n_0$". Then there is a rapid introduction to the symbolism used in logic. After this the basics of the elementary theory of sets are reviewed, including a discussion of the notion of equivalence relation, and of the important fact that an equivalence relation defined on a set, determines a partition of the set into equivalence classes.

Chapter 3 introduces the notion of incidence geometry as a set together with a collection formed by some of its subsets, having three properties called *axioms of incidence*. Then examples of incidence geometries of various kinds are presented. Then the "main" examples of incidence geometries, namely the real cartesian plane, the hyperbolic plane and the elliptic geometry, are presented in complete detail. In particular, complete proofs, *based only* on the properties of addition and multiplication in the real number system, that the three axioms of incidence hold in these examples, are supplied. After this the subject of parallelism of lines is discussed, and an additional axiom, called Playfair's axiom is studied. Playfair's axiom is a refined version of Euclid's fifth postulate. The chapter ends with a long discussion of how paralllelism behaves in all the examples of incidence geometries previously given. It is of particular importance the discussion of the behaviour of parallelism in the real cartesian plane, the hyperbolic plane and the elliptic geometry. It is shown that in the real cartesian plane, given any point A and any line l, with A not in l, *there exists a unique line* passing through A and being parallel to l; that in the hyperbolic plane, given any point A and any line l, with A not in l, *there exists an infinite number of lines* passing through A and being parallel to l; and that in the elliptic geometry, given any point A and any line l, with A not in l, *there is no line* passing through A and being parallel to l.

Chapter 4 treats the formalization of the notion that one point is between two other points, i.e., the concept of a betweenness structure for an incidence geometry. A betweenness structure is defined as a collection of ordered triples of points of the plane, having four properties called the *betweenness axioms.* We remark that the realization by Hilbert that one of the main deficiencies of Euclid's axiomatics, making some of Euclid's proofs in the Elements ultimate dependant on pictures, lied in the lack of axioms governing betweenness, constitutes perhaps his main contribution for saving Euclid's work. The chapter begins with the definition of betweenness structure for an incidence geometry. This structure makes it possible to define segments, triangles and the convexity of a subset of the plane. Then it is shown that a line l divides the plane minus l into two parts, called sides of the plane divided by l; and also that a point A of a line l, divides l minus $\{A\}$ into two parts, called the sides of l divided by A. As consequences the following interesting facts are proved to hold in any incidence geometry equipped with a betweenness structure, namely, that the endpoints of a segment are entirely determined by the segment, that each line is formed by an infinite number of points, and that there is a point between any two given points. Next, the important notion of ray is introduced, and a long theorem containing a bunch of facts about rays which prove very useful for the rest of the book, is stated and proved. Then the fundamental notion of angle is introduced, followed by a discussion of the important notion of interior of an angle. An important result called Crossbar Theorem is then presented at length. The chapter ends with a discussion of the usual betweenness structures carried by the real cartesian plane and the hyperbolic plane. Elliptic geometry is abandoned at this point for the rest of the book, due to the fact that it does not admit any betweenness structure. It does admit a *modified* betweenness structure though (see [2]).

Chapter 5 introduces the notion of structure of congruence of segments for an incidence geometry equipped with a betweenness structure. It is defined as a collection of ordered pairs of segments, having three properties called *axioms of congruence of segments*. Then a useful result, called subtraction of segments, is proven. Next, the notion that a segment is less than another segment, is introduced, and its main properties are stated and proven, using subtraction of segments as the main tool. The rest of the chapter is devoted to define the usual congruence of segment structures in the real cartesian plane and in the hyperbolic plane, and proving that these satisfy the three axioms.

Chapter 6 is dedicated to the notion of a structure of congruence of angles for an incidence geometry equipped with a betweennes structure and a congruence of segments structure. It is defined as a collection of ordered pairs of angles, having three properties called *axioms of congruence of angles*. This structure allows for the definition of congruence of triangles. Then the concepts of adjacent angles, supplementary angles and vertical angles are defined. The important results summarized as "angles which are supplementary of congruent angles, are congruent", "a pair of adjacent angles which are congruent to a pair of supplementary angles, are supplementary" and "vertical angles are congruent", are precisely stated and proved. Then the *angle addition theorem* and the *angle subtraction theorem* are discussed. The notion of an angle being less than another angle is introduced, and its main properties are proven. Right angles are then defined as angles which are congruent to any of its (two) supplementary angles, and the congruence of any two right angles is established. The study of the usual structures of congruence of angles for the real cartesian plane and the hyperbolic plane occupy the rest of the chapter.

Finally, Chapter 7 is dedicated to Hilbert Planes. Hilbert Planes are incidence geometries equipped with a betweenness structure, a congruence of segments structure and a congruence of angles structure. This requires that a total of thirteen axioms are satisfied, three incidence axioms, four betweenness axioms, three congruence of segments axioms and three congruence of angles axioms. The main examples of Hilbert Planes are the real cartesian plane and the hyperbolic plane. The rest of the chapter is dedicated to the study of Book I of Euclid's Elements, a la Hilbert.

1.5 How to study this book

Your attitude, in order to really grasp the material, should be that of a hyperactive student. Leisurely studying the material will not do it! Read every sentence carefully. Read the examples and do as many exercises as possible. You may even try to create some exercises. In a first reading of the book, you may skip certain particularly long examples, like the one on parallelism in the hyperbolic plane.